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2N+1 highest amplitude of the modulus of the N-th order AP breather and other 2N-2 parameters solutions to the NLS equation.

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Abstract

We construct here new deformations of the AP breather (Akhmediev-Peregrine breather) of order N (or AP_N breather) with $2N-2$ real parameters. Other families of quasi-rational solutions of the NLS equation are obtained. We evaluate the highest amplitude of the modulus of AP breather of order N; we give the proof that the highest amplitude of the AP_N breather is equal to $2N+1$. We get new formulas for the solutions of the NLS equation, different from these already given in previous works. New solutions for the order 8 and their deformations according to the parameters are explicitly given. We get the triangular configurations as well as isolated rings at the same time. Moreover, the appearance for certain values of the parameters, of new configurations of concentric rings are underscored.

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1 Introduction

We consider the one dimensional focusing nonlinear Schrödinger equation (NLS) to describe the phenomena of rogue waves. We recall that the term of rogue or freak wave was first introduced in the scientific community by Draper in 1964 [7].
The rogue waves phenomenon plays actually a significant role in other fields; in nonlinear optics [36], Bose-Einstein condensate [5], atmosphere [37] and even finance [38].
The first results concerning the NLS equation date from the works of Zakharov and Shabat in 1972 who solved it using the inverse scattering method [39, 40]. Its and Kotlyarov first constructed periodic and almost periodic algebro-geometric solutions to the focusing NLS equation in 1976 [29, 30]. It is in 1979 that Ma found the first breather type solution of the NLS equation [33]. In 1983, the first quasi rational solution of NLS equation was constructed by Peregrine [35]. Akhmediev, Eleonski and Kulagin obtained for the first time the second order rational solution and predicted the existence of an infinite hierarchy of higher-order rational solutions [2, 3]. Other analogues of the AP breathers of order 3 and 4 were constructed using Darboux transformations, in a series of articles by Akhmediev et al. [1, 4, 6].
Recently, many works about NLS equation have been published using different methods. Rational solutions of the NLS equation has been written as a quotient of two wronskians in 2010 [8]. One year after, the present author constructed in [9] an other representation of the solutions of the NLS equation in terms of a ratio of two wronskians determinants of even order $2N$ composed of elementary functions using truncated Riemann theta functions; rational solutions were obtained when one of the parameters tends towards 0. Guo, Ling and Liu found in 2012 an other representation of the solutions as a ratio of two determinants [27] using generalized Darboux transform. A new approach has been done by Ohta and Yang in [34] using Hirota bilinear method. In the same year, the present author obtained rational solutions in terms of determinants which does not involve limits in [12]. In 2013, we have constructed explicitly deformations of AP breather of order N depending on $2N-2$ parameters by giving their expressions in terms of quotient of polynomials in x and t for the orders until $N=7$, as given for example in [16].
The present paper gives here other new multi-parametric families of quasi rational solutions of NLS of order N in terms of determinants (determinants of order $2N$) dependent on $2N-2$ real parameters. New solutions different

from all the previous one are obtained. With this representation, one recovers at the same time the ring or concentric rings structure and the triangular shapes also found by Ohta and Yang [34], Akhmediev et al. [31].
We construct solutions depending on $2N-2$ parameters which give the AP breather as particular case when all the parameters are equal to 0 : for this reason, we will call these solutions, $2N-2$ parameters deformations of the AP_N breather.
The paper is organized as follows. We construct new quasi rational solutions depending a priori on $2N-2$ parameters at the order N. Then we prove that the highest amplitude of the modulus of the AP breather of order N is equal to $2N+1$. We construct the AP breathers for $N=1$ to $N=10$; we only give the corresponding plots of the modulus in the $(x;t)$ plane. After, one constructs various figures to illustrate the evolution of the solutions according to the parameters for the order 8. One obtains at the same time triangular configurations and ring structures.

2 New wronskian representation of solutions of NLS equation

We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0. \tag{1}$$

We recall the main result obtained in [9].

Theorem 2.1 *The function v defined by*

$$v(x,t) = \frac{W_3(0)}{W_1(0)} \exp(2it - i\varphi) \tag{2}$$

is solution of the NLS equation (1).

In (2), $W_r(y) = W(\phi_{r,1}, \ldots, \phi_{r,2N})$ is the wronskian of order $2N$

$$W_r(y) = \det[(\partial_y^{\mu-1}\phi_{r,\nu})_{\nu,\,\mu\in[1,\ldots,2N]}], \quad r=1,\, r=3. \tag{3}$$

The functions $\phi_{r,\nu}$ are defined by

$$\begin{aligned}
&\phi_{r,\nu}(y) = \sin\Theta_{r,\nu}, \quad 1 \le \nu \le N,\\
&\phi_{r,\nu}(y) = \cos\Theta_{r,\nu}, \quad N+1 \le \nu \le 2N,\\
&\Theta_{r,\nu} = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu, \quad 1 \le \nu \le 2N.
\end{aligned} \tag{4}$$

The terms κ_ν, δ_ν, γ_ν are functions of the parameters λ_ν, $\nu = 1, \ldots, 2N$ satisfying the relations

$$0 < \lambda_j < 1, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \leq j \leq N. \tag{5}$$

They are given by the following equations,

$$\begin{array}{l} \kappa_j = 2\sqrt{1-\lambda_j^2}, \quad \delta_j = \kappa_j \lambda_j, \quad \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}, \\ \kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = 1/\gamma_j, \quad j = 1 \ldots N. \end{array} \tag{6}$$

The terms $x_{r,\nu}$, $(r = 3,\, 1)$ are defined by

$$x_{r,\nu} = (r-1) \ln \tfrac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1 \leq j \leq 2N. \tag{7}$$

The parameters e_ν are given by

$$e_j = i a_j - b_j, \quad e_{N+j} = i a_j + b_j, \quad 1 \leq j \leq N, \tag{8}$$

where a_j and b_j, for $1 \leq j \leq N$ are arbitrary real numbers.
We choose here to give a new representation of the solutions of the NLS equation depending only on terms γ_ν, $1 \leq \nu \leq 2N$. From the relations (6), (7), we can express the terms κ_ν, δ_ν and $x_{r,\nu}$ in function of γ_ν, for $1 \leq \nu \leq 2N$ and we obtain :

$$\begin{array}{l} \kappa_j = \frac{4\gamma_j}{(1+\gamma_j^2)}, \quad \delta_j = \frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}, \quad x_{r,j} = (r-1)\ln\frac{\gamma_j - i}{\gamma_j + i}, \quad 1 \leq j \leq N, \\ \kappa_j = \frac{4\gamma_j}{(1+\gamma_j^2)}, \quad \delta_j = -\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}, \quad x_{r,j} = (r-1)\ln\frac{\gamma_j + i}{\gamma_j - i}, \quad N+1 \leq j \leq 2N. \end{array} \tag{9}$$

We have the following new representation :

Theorem 2.2 *The function v defined by*

$$v(x,t) = \frac{\det[(\partial_y^{\mu-1}\tilde{\phi}_{3,\nu}(0))_{\nu,\,\mu\in[1,\ldots,2N]}]}{\det[(\partial_y^{\mu-1}\tilde{\phi}_{1,\nu}(0))_{\nu,\,\mu\in[1,\ldots,2N]}]}\exp(2it - i\varphi) \tag{10}$$

is solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0.$$

The functions $\tilde{\phi}_{r,\nu}$ *are defined by*

$$\tilde{\phi}_{r,j}(y) = \sin\left(\frac{2\gamma_j}{(1+\gamma_j^2)}x + i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - i\frac{(r-1)}{2}\ln\frac{\gamma_j - i}{\gamma_j + i} + \gamma_j y - ie_j\right), \quad 1 \leq j \leq N,$$
$$\tilde{\phi}_{r,N+j}(y) = \cos\left(\frac{2\gamma_j}{(1+\gamma_j^2)}x - i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t + i\frac{(r-1)}{2}\ln\frac{\gamma_j - i}{\gamma_j + i} + \frac{1}{\gamma_j}y - ie_{N+j}\right), \quad 1 \leq j \leq N,$$

where $\gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}$, $1 \leq j \leq N$. (11)

λ_j *is an arbitrary real parameter such that* $0 < \lambda_j < 1$, $\lambda_{N+j} = -\lambda_j$, $1 \leq j \leq N$.
The terms e_ν *are defined by* $e_j = ia_j - b_j$, $e_{N+j} = ia_j + b_j$, $1 \leq j \leq N$,
where a_j *and* b_j *are arbitrary real numbers*, $1 \leq j \leq N$.

Remark 2.1 *In the formula (10), the determinants* $\det[(\partial_y^{\mu-1} f_\nu(0))_{\nu,\,\mu\in[1,\ldots,2N]}]$ *are the wronskians of the functions* $f_1, \ldots, f_{2N}$ *evaluated in* $y = 0$. *In particular* $\partial_y^0 f_\nu$ *means* f_ν.

3 Families of quasi-rational solutions depending on $2N-2$ parameters of NLS equation in terms of a ratio of two determinants of order N.

In the following, to get quasi-rational solutions of the NLS equation, we have to take the limits $\lambda_1 \to 1$ for $1 \leq j \leq N$ and $\lambda_1 \to -1$ for $N+1 \leq j \leq 2N$. For this, we choose $\lambda_j = 1 - 2j\epsilon^2$. When ϵ goes to 0, we realize limited expansions at order M of all the functions $\Phi_{r,\nu}$.
We use the following notations :

$$X_j = \frac{2\gamma_j}{(1+\gamma_j^2)}x + i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - i\ln\frac{\gamma_j - i}{\gamma_j + i} - ie_j,$$
$$X_{N+j} = \frac{2\gamma_j}{(1+\gamma_j^2)}x - i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t + i\ln\frac{\gamma_j - i}{\gamma_j + i} - ie_{N+j},$$
for $1 \leq j \leq N$.
$$Y_j = \frac{2\gamma_j}{(1+\gamma_j^2)}x + i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - ie_j, \tag{12}$$
$$Y_{N+j} = \frac{2\gamma_j}{(1+\gamma_j^2)}x - i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - ie_{N+j},$$
for $1 \leq j \leq N$.

The terms γ_ν and e_ν are defined by (11). We have any freedom to choose the terms a_j and b_j. This is the crucial point. We choose a_j and b_j in the form

$$a_j = \sum_{k=1}^{N-1} \tilde{a_k} j^{2k+1} \epsilon^{2k+1}, \quad b_j = \sum_{k=1}^{N-1} \tilde{b_k} j^{2k+1} \epsilon^{2k+1}, \quad 1 \leq j \leq N. \tag{13}$$

In order to rewrite the quotient of wronskians defined in (11), we use the following functions :

$$\begin{array}{l}
\varphi_{4j+1,k} = \gamma_k^{4j-1} \sin X_k, \quad \varphi_{4j+2,k} = \gamma_k^{4j} \cos X_k, \\
\varphi_{4j+3,k} = -\gamma_k^{4j+1} \sin X_k, \quad \varphi_{4j+4,k} = -\gamma_k^{4j+2} \cos X_k, \\
\varphi_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos X_{N+k}, \quad \varphi_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin X_{N+k}, \\
\varphi_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos X_{N+k}, \quad \varphi_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin X_{N+k}, \quad 1 \leq k \leq N.
\end{array} \tag{14}$$

We define the functions $g_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way, we replace only the term X_k by Y_k.

$$\begin{array}{l}
\psi_{4j+1,k} = \gamma_k^{4j-1} \sin Y_k, \quad \psi_{4j+2,k} = \gamma_k^{4j} \cos Y_k, \\
\psi_{4j+3,k} = -\gamma_k^{4j+1} \sin Y_k, \quad \psi_{4j+4,k} = -\gamma_k^{4j+2} \cos Y_k, \\
\psi_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos Y_{N+k}, \quad \psi_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin Y_{N+k}, \\
\psi_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos Y_{N+k}, \quad \psi_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin Y_{N+k}, \quad 1 \leq k \leq N
\end{array} \tag{15}$$

The quotient of wronskians $q(x,t)$ defined by

$$q(x,t) := \frac{\det[(\partial_y^{\mu-1} \tilde{\phi}_{3,\nu}(0))_{\nu,\, \mu \in [1,\ldots,2N]}]}{\det[(\partial_y^{\mu-1} \tilde{\phi}_{1,\nu}(0))_{\nu,\, \mu \in [1,\ldots,2N]}]}$$

can be written as

$$q(x,t) = \frac{\Delta_3}{\Delta_1} = \frac{\det(\varphi_{j,k})_{j,\, k \in [1,2N]}}{\det(\psi_{j,k})_{j,\, k \in [1,2N]}}. \tag{16}$$

All the functions $\varphi_{j,k}$ and $\psi_{j,k}$ and their derivatives depend on ϵ and can all be prolonged by continuity when $\epsilon = 0$.
Then we use the expansions

$$\begin{array}{l}
\varphi_{j,k}(x,t,\epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!} \varphi_{j,1}[l] k^{2l} \epsilon^{2l} + O(\epsilon^{2N}), \quad \varphi_{j,1}[l] = \frac{\partial^{2l} \varphi_{j,1}}{\partial \epsilon^{2l}}(x,t,0), \\
\varphi_{j,1}[0] = \varphi_{j,1}(x,t,0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1, \\
\varphi_{j,N+k}(x,t,\epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!} \varphi_{j,N+1}[l] k^{2l} \epsilon^{2l} + O(\epsilon^{2N}), \quad \varphi_{j,N+1}[l] = \frac{\partial^{2l} \varphi_{j,N+1}}{\partial \epsilon^{2l}}(x,t,0), \\
\varphi_{j,N+1}[0] = \varphi_{j,N+1}(x,t,0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1.
\end{array} \tag{17}$$

We have the same expansions for the functions $\psi_{j,k}$.

$$\begin{array}{l} \psi_{j,k}(x,t,\epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!}\psi_{j,1}[l]k^{2l}\epsilon^{2l} + O(\epsilon^{2N}), \quad \psi_{j,1}[l] = \frac{\partial^{2l}\psi_{j,1}}{\partial\epsilon^{2l}}(x,t,0), \\ \psi_{j,1}[0] = \psi_{j,1}(x,t,0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1, \\ \psi_{j,N+k}(x,t,\epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!}\psi_{j,N+1}[l]k^{2l}\epsilon^{2l} + O(\epsilon^{2N}), \quad \psi_{j,N+1}[l] = \frac{\partial^{2l}\psi_{j,N+1}}{\partial\epsilon^{2l}}(x,t,0), \\ \psi_{j,N+1}[0] = \psi_{j,N+1}(x,t,0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad N+1 \leq k \leq 2N. \end{array} \tag{18}$$

Then we get the following result :

Theorem 3.1 *The function v defined by*

$$v(x,t) = \exp(2it - i\varphi) \times \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} \tag{19}$$

is a quasi-rational solution of the NLS equation (1) $iv_t + v_{xx} + 2|v|^2v = 0$ depending on $2N-2$ real parameters $\tilde{a}_j$, $\tilde{b}_j$, $1 \leq j \leq N-1$, where

$$\begin{array}{l} n_{j1} = \varphi_{j,1}(x,t,0), \ 1 \leq j \leq 2N \quad n_{jk} = \frac{\partial^{2k-2}\varphi_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0), \\ n_{jN+1} = \varphi_{j,N+1}(x,t,0), \ 1 \leq j \leq 2N \quad n_{jN+k} = \frac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0), \\ d_{j1} = \psi_{j,1}(x,t,0), \ 1 \leq j \leq 2N \quad d_{jk} = \frac{\partial^{2k-2}\psi_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0), \\ d_{jN+1} = \psi_{j,N+1}(x,t,0), \ 1 \leq j \leq 2N \quad d_{jN+k} = \frac{\partial^{2k-2}\psi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0), \\ 2 \leq k \leq N, \ 1 \leq j \leq 2N \end{array}$$

The functions φ and ψ are defined in (14), (15).

Proof : We eliminate in each column k (and $N+k$) of the determinants appearing in $q(x,t)$, the powers of ϵ strictly lower than $2(k-1)$ by combining the columns of them successively. We begin by the components j of the columns 1 and $N+1$; they are respectively equal by definition to $\varphi_{j1}[0]+0(\epsilon)$ for C_1, $\varphi_{jN+1}[0]+0(\epsilon)$ for C_{N+1} of Δ_3, and $\psi_{j1}[0]+0(\epsilon)$ for C'_1, $\psi_{jN+1}[0]+0(\epsilon)$ for C'_{N+1} of Δ_1.
At the first step of the reduction, we replace the columns C_k by $C_k - C_1$ and C_{N+k} by $C_{N+k} - C_{N+1}$ for $2 \leq k \leq N$, for Δ_3; we do the same changes for Δ_1. Each component j of the column C_k of Δ_3 can be rewritten as $\sum_{l=1}^{N-1} \frac{1}{(2l)!}\varphi_{j,1}[l](k^{2l}-1)\epsilon^{2l}$ and the column C_{N+k} replaced by $\sum_{l=1}^{N-1} \frac{1}{(2l)!}\varphi_{j,N+1}[l](k^{2l}-1)\epsilon^{2l}$ for $2 \leq k \leq N$. For Δ_1, we have the same reductions, each component j of the column C'_k of can be rewritten as $\sum_{l=1}^{N-1} \frac{1}{(2l)!}\psi_{j,1}[l](k^{2l}-1)\epsilon^{2l}$ and the column C'_{N+k} replaced by $\sum_{l=1}^{N-1} \frac{1}{(2l)!}\psi_{j,N+1}[l](k^{2l}-1)\epsilon^{2l}$ for $2 \leq k \leq N$.

We can factorize in Δ_3 and Δ_1 in each column k and $N+k$ the term $\frac{k^2-1}{2}\epsilon^2$ for $2 \leq k \leq N$, and so simplify these common terms in numerator and denominator.
If we restrict the developments at order 1 in columns 2 and $N+2$, we get respectively $\varphi_{j1}[1] + 0(\epsilon)$ for the component j of C_2, $\varphi_{jN+1}[1] + 0(\epsilon)$ for the component j of C_{N+2} of Δ_3, and $\psi_{j1}[1] + 0(\epsilon)$ for the component j of C'_2, $\psi_{jN+1}[1] + 0(\epsilon)$ for the component j of C'_{N+2} of Δ_1. This algorithm can be continued until the columns C_N, C_{2N} of Δ_3 and C'_N, C'_{2N} of Δ_1.
Then taking the limit when ϵ tends to 0, $q(x,t)$ can be replaced by $Q(x,t)$ defined by :

$$Q(x,t) := \frac{\begin{vmatrix} \varphi_{1,1}[0] & \dots & \varphi_{1,1}[N-1] & \varphi_{1,N+1}[0] & \dots & \varphi_{1,N+1}[N-1] \\ \varphi_{2,1}[0] & \dots & \varphi_{2,1}[N-1] & \varphi_{2,N+1}[0] & \dots & \varphi_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_{2N,1}[0] & \dots & \varphi_{2N,1}[N-1] & \varphi_{2N,N+1}[0] & \dots & \varphi_{2N,N+1}[N-1] \end{vmatrix}}{\begin{vmatrix} \psi_{1,1}[0] & \dots & \psi_{1,1}[N-1] & \psi_{1,N+1}[0] & \dots & \psi_{1,N+1}[N-1] \\ \psi_{2,1}[0] & \dots & \psi_{2,1}[N-1] & \psi_{2,N+1}[0] & \dots & \psi_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{2N,1}[0] & \dots & \psi_{2N,1}[N-1] & \psi_{2N,N+1}[0] & \dots & \psi_{2N,N+1}[N-1] \end{vmatrix}} \tag{20}$$

So the solution of the NLS equation takes the form :

$$v(x,t) = \exp(2it - i\varphi) \times Q(x,t)$$

So the obtain the result given in (37). $\square$

4 The AP breather of order N and its highest amplitude of the modulus equal to $2N+1$

There is any freedom to choose γ_j in such a way that the conditions on λ_j are checked. We know from previous works [9, 12] that the AP breathers are obtained when all the parameters $\tilde{a}_j$ and $\tilde{b}_j$ are equal to 0. In order to get the more simple expressions in the determinants, we choose particular solutions in the previous families.
Here we choose $\gamma_j = j\epsilon$ as simple as possible in order to have the conditions on λ_j checked.

Theorem 4.1 *The function v_0 defined by*

$$v_0(x,t) = \exp(2it - i\varphi) \times \left(\frac{\det((n_{jk})_{j,k \in [1,2N]})}{\det((d_{jk})_{j,k \in [1,2N]})} \right)_{(\tilde{a_j} = \tilde{b_j} = 0,\, 1 \leq j \leq N-1)} \tag{21}$$

is the AP breather of order N solution of the NLS equation (1) whose highest amplitude in modulus is equal to $2N+1$.

Remark 4.1 *In (21), the matrices $(n_{jk})_{j,k \in [1,2N]}$ and $(d_{jk})_{j,k \in [1,2N]}$ are defined in (20).*

Proof : We recall that AP_N is the AP breather of order N. All the previous analytical and numerical studies ([9, 12, 15]) shows that the maximum of AP_N is reached for $(x,t) = (0,0)$. We are going to establish that the value of the modulus of this solution AP_N denoted $v_0(0,0)$ is equal to $2N+1$.
We need to analyze the functions $\varphi_{k,1}$, $\varphi_{k,N+1}$ and $\psi_{k,1}$, $\psi_{k,N+1}$.
We denote $(l_{kj})_{k,j \in [1,2N]}$ the matrix defined by

$$l_{kj} = \frac{\partial^{2j-2}}{\partial \epsilon^{2j-2}} \varphi_{k1}(0,0), \quad l_{k,j+N} = \frac{\partial^{2j-2}}{\partial \epsilon^{2j-2}} \varphi_{k,1+N}(0,0), \quad 1 \leq j \leq N,\, 1 \leq k \leq 2N,$$

and $(l'_{kj})_{k,j \in [1,2N]}$ the matrix defined by

$$l'_{kj} = \frac{\partial^{2j-2}}{\partial \epsilon^{2j-2}} \psi_{k1}(0,0), \quad l_{k,j+N} = \frac{\partial^{2j-2}}{\partial \epsilon^{2j-2}} \psi_{k,1+N}(0,0), \quad 1 \leq j \leq N,\, 1 \leq k \leq 2N,$$

$\frac{\partial^0}{\partial x^0}\varphi$ meaning φ. We remark that with these notations, the matrix $(l_{kj})_{k,j \in [1,2N]}$ evaluated in $\epsilon = 0$ is exactly $((n_{kj})_{\tilde{a_j} = \tilde{b_j} = 0,\, 1 \leq j \leq N-1, x=0, t=0})_{k,j \in [1,2N]}$ and the matrix $(l'_{kj})_{k,j \in [1,2N]}$ evaluated in $\epsilon = 0$ is exactly $((d_{kj})_{\tilde{a_j} = \tilde{b_j} = 0,\, 1 \leq j \leq N-1, x=0, t=0})_{k,j \in [1,2N]}$, defined in (20). We don't change the value of the quotient of the determinants in the solution v if we replace $x_{3,j} = 2 \ln \frac{\gamma_j - i}{\gamma_j + 1}$ by $2 \ln \frac{1 + i\gamma_j}{1 - i\gamma_j}$, because the terms $-ix_{3,j}$ change in $-ix_{3,j} + 2\pi$.
We have four cases to study depending on the parity of k.
1. We first study l_{kj}.
a. l_{k1} for k odd, $k = 2s+1$, for $x = 0$ and $t = 0$.

$$l_{k1} = (-1)^s \sin(-i \ln \frac{1 + i\epsilon}{1 - i\epsilon}) \epsilon^{k-2} = (-1)^s \frac{2\epsilon^{2s}}{1 + \epsilon^2}$$

$$= \sum_{t=s}^{N+s} (-1)^t 2\epsilon^{2t} + O(\epsilon^{2N+2s}) = \sum_{t=0}^{N} \frac{1}{(2t)!} \frac{\partial^{2t} \varphi_{k1}}{\partial \epsilon^{2t}} \epsilon^{2t} + O(\epsilon^{2N+1}).$$

So we get : for $t < s$, $\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}} = 0$ and for $t \geq s$, $\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}} = (-1)^t 2$. It can be rewritten as

Proposition 4.1

$$\begin{gathered} 0 \leq t \leq N-1,\ 0 \leq s \leq N-1, \\ n_{2s+1,t+1} = 0 \ \textit{if}\ t < s, \quad n_{2s+1,t+1} = (-1)^t 2 \ \textit{if}\ t \geq s. \end{gathered} \tag{22}$$

b. l_{k1} for k even, $k = 2s$.

$$l_{k1} = (-1)^{s+1}\cos(-i\ln\frac{1+i\epsilon}{1-i\epsilon})\epsilon^{k-2} = (-1)^{s+1}\frac{\epsilon^{2s-2}(1-\epsilon^2)}{1+\epsilon^2}$$

$$= (-1)^{s-1}2\epsilon^{2(s-1)} + \sum_{t=s}^{N+s}(-1)^t 2\epsilon^{2t} + O(\epsilon^{2N+2s+1}) = \sum_{t=0}^{N}\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}}\epsilon^{2t} + O(\epsilon^{2N+1}).$$

So we get : for $t < s-1$, $\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}} = 0$; for $t = s-1$, $\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}} = (-1)^{s-1}$; for $t > s-1$, $\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}} = (-1)^t 2$. It can be rewritten as

Proposition 4.2

$$\begin{gathered} 0 \leq t \leq N-1,\ 1 \leq s \leq N, \quad n_{2s,t+1} = 0 \ \textit{if}\ t < s-1, \\ n_{2s,t+1} = (-1)^{s-1} \ \textit{if}\ t = s-1, \quad n_{2s,t+1} = (-1)^t 2 \ \textit{if}\ t > s-1. \end{gathered} \tag{23}$$

c. l_{kN+1} for k odd, $k = 2s+1$.

$$l_{k,N+1} = (-1)^s\cos(i\ln\frac{1+i\epsilon}{1-i\epsilon})\epsilon^{2N-k-1} = (-1)^s\frac{\epsilon^{2N-2s-2}(1-\epsilon^2)}{1+\epsilon^2}$$

$$= (-1)^s\epsilon^{2(N-s-1)} + \sum_{t=N-s}^{2N-s}(-1)^{t+N+1}2\epsilon^{2t} + O(\epsilon^{4N-2s+1}) = \sum_{t=0}^{N}\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}}\epsilon^{2t} + O(\epsilon^{2N+1}).$$

So we get : for $t < N-s-1$, $\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}} = 0$; for $t = N-s-1$, $\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}} = (-1)^s$; for $t > N-s-1$, $\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}} = (-1)^{t+N+1}2$. It can be rewritten as

Proposition 4.3

$$\begin{gathered} 0 \leq t \leq N-1,\ 0 \leq s \leq N-1, \quad n_{2s+1,N+1+t} = 0 \ \textit{if}\ t < N-s-1, \\ n_{2s+1,N+1+t} = (-1)^s \ \textit{if}\ t = N-s-1, \quad n_{2s+1,N+1+t} = (-1)^{t+N+1}2 \ \textit{if}\ t > N-s-1. \end{gathered} \tag{24}$$

d. $l_{k,N+1}$ for k even, $k = 2s$.

$$l_{k1} = (-1)^{s+1}\sin(-i\ln\frac{1+i\epsilon}{1-i\epsilon})\epsilon^{2N-k-1} = (-1)^{s+1}\frac{2\epsilon^{2N-2s}}{1+\epsilon^2}$$

$$= \sum_{t=N-s}^{2N-s}(-1)^{t-N+1}2\epsilon^{2t} + O(\epsilon^{4N-2s+1}) = \sum_{t=0}^{N}\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}}\epsilon^{2t} + O(\epsilon^{2N+1}).$$

So we get : for $t < N-s$, $\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}} = 0$; for $t \geq N-s$, $\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}} = (-1)^{t+N+1}2$. It can be rewritten as

Proposition 4.4

$$0 \leq t \leq N-1,\ 1 \leq s \leq N,$$
$$n_{2s,N+t+1} = 0 \ \textit{if}\ t < N-s, \quad n_{2s,N+t+1} = (-1)^{t+N+1}2 \ \textit{if}\ t \geq N-s. \quad (25)$$

Then $A_3(N) := (\prod_{l=1}^{N-1}(2l)!)^{-2}\det((n_{jk})_{j,k\in[1,2N]})_{(\tilde{a_j}=\tilde{b_j}=0,\,1\leq j\leq N-1,\,x=0,\,t=0)}$ can be written as $\det((\tilde{n}_{ij})_{i,j\in[1,2N]})$ defined by

$$\begin{vmatrix}
(-1)^0\times 2 & (-1)^1\times 2 & \dots & (-1)^{N-2}\times 2 & (-1)^{N-1}\times 2 & 0 & 0 & \dots & 0 & (-1)^0 \\
(-1)^0 & (-1)^1\times 2 & \dots & (-1)^{N-2}\times 2 & (-1)^{N-1}\times 2 & 0 & 0 & \dots & 0 & (-1)^0\times 2 \\
0 & (-1)^1\times 2 & \dots & (-1)^{N-2}\times 2 & (-1)^{N-1}\times 2 & 0 & 0 & \dots & (-1)^1 & (-1)^0\times 2 \\
0 & (-1)^1 & \dots & (-1)^{N-1}\times 2 & (-1)^{N-1}\times 2 & 0 & 0 & \dots & (-1)^1\times 2 & (-1)^0\times 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & (-1)^{N-2}\times 2 & (-1)^{N-1}\times 2 & 0 & (-1)^{N-2} & \dots & (-1)^1\times 2 & (-1)^0\times 2 \\
0 & 0 & \dots & (-1)^{N-2} & (-1)^{N-1}\times 2 & 0 & (-1)^{N-2}\times 2 & \dots & (-1)^1\times 2 & (-1)^0\times 2 \\
0 & 0 & \dots & 0 & (-1)^{N-1}\times 2 & (-1)^{N-1} & (-1)^{N-2}\times 2 & \dots & (-1)^1\times 2 & (-1)^0\times 2 \\
0 & 0 & \dots & 0 & (-1)^{N-1} & (-1)^{N-1}\times 2 & (-1)^{N-2}\times 2 & \dots & (-1)^1\times 2 & (-1)^0\times 2
\end{vmatrix} \quad (26)$$

Then we first factorize in each column j the term $(-1)^{j-1}$ for $1 \leq j \leq N$ and $(-1)^{N-j}$ for $N+1 \leq N+j \leq 2N$; the common factor is $(-1)^{N(N-1)}$ equal to 1. We get the following determinant

$$A_3(N) = \begin{vmatrix}
2 & 2 & \dots & 2 & 2 & 0 & 0 & \dots & 0 & 1 \\
1 & 2 & \dots & 2 & 2 & 0 & 0 & \dots & 0 & 2 \\
0 & 2 & \dots & 2 & 2 & 0 & 0 & \dots & 1 & 2 \\
0 & 1 & \dots & 2 & 2 & 0 & 0 & \dots & 2 & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 2 & 2 & 0 & 1 & \dots & 2 & 2 \\
0 & 0 & \dots & 1 & 2 & 0 & 2 & \dots & 2 & 2 \\
0 & 0 & \dots & 0 & 2 & 1 & 2 & \dots & 2 & 2 \\
0 & 0 & \dots & 0 & 1 & 2 & 2 & \dots & 2 & 2
\end{vmatrix} \quad (27)$$

Then we realize the following transformations on the rows L_j : we replace L_j by $L_j - L_{j+1}$ for $1 \leq j \leq M-1$. We get

$$A_3(N) = \begin{vmatrix}
1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\
1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\
0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\
0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 1 & 0 & 0 & -1 & \dots & 0 & 0 \\
0 & 0 & \dots & 1 & 0 & -1 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 1 & -1 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 1 & 2 & 2 & \dots & 2 & 2
\end{vmatrix} \quad (28)$$

We expand along the first row to obtain

$$A_3(N) = \begin{vmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 2 & 2 & \dots & 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 2 & 2 & \dots & 2 & 2 \end{vmatrix} \tag{29}$$

We expand the first determinant along the last column and the second one along the first column to obtain

$$A_3(N) = 2\begin{vmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & -1 & 0 & \dots & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 2 & 2 & \dots & 2 & 2 \end{vmatrix} \tag{30}$$

In this last relation the second determinant is nothing else but $A_3(N-1)$. Thus $A_3(N)$ can be written as $2\Delta_3(N-1)+A_3(N-1)$. We have to calculate $\Delta_3(N-1)$. We can expand this determinant first along the first row, then again along the first row of the new determinant. We get :

$$\Delta_3(N-1) = \begin{vmatrix} 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & -1 & 0 & \dots & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & -1 & 0 & \dots & 0 & 0 \end{vmatrix} = \Delta_3(N-2) \tag{31}$$

So we get $\Delta_3(N-1) = \Delta_3(N-2)$; as $\Delta_3(1) = 1$, we clearly have $\Delta_3(N) = 1$, for each integer $N \geq 1$.
Thus we have the following recurrence relation $A_3(N) = 2\Delta_3(N-1)+A_3(N-1) = A_3(N-1)+2$; as $A_3(1) = 3$, we get $A_3(N) = A_3(N-1)+2 = A_3(1)+2(N-1) = 2N+1$.

2. Then we study the elements l'_{kj} of the denominator of v_0.
a. l'_{k1} for k odd, $k = 2s+1$, for $x = 0$ and $t = 0$.

$$l'_{k1} = (-1)^s \sin(0) = 0$$

So we get

Proposition 4.5

$$
\begin{aligned}
0 \le t \le N-1,\ 0 \le s \le N-1,& \\
n_{2s+1,t+1} = 0.& \qquad (32)
\end{aligned}
$$

b. l'_{k1} for k even, $k = 2s$.

$$
\begin{aligned}
l'_{k1} &= (-1)^{s+1}\cos(0) = (-1)^{s+1}\epsilon^{2s-2} \\
&= \sum_{t=0}^{N} \frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}}\epsilon^{2t} + O(\epsilon^{2N+1}).
\end{aligned}
$$

So we get : for $t \neq s-1$, $\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}} = 0$; for $t = s-1$, $\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}} = (-1)^{s+1}$. It can be rewritten as

Proposition 4.6

$$
\begin{aligned}
&0 \le t \le N-1,\ 1 \le s \le N, \\
n_{2s,t+1} = 0 \ \textit{if}\ t \neq s-1, \quad &n_{2s,t+1} = (-1)^{s+1} \ \textit{if}\ t = s-1. \qquad (33)
\end{aligned}
$$

c. l'_{kN+1} for k odd, $k = 2s+1$.

$$
\begin{aligned}
l'_{k,N+1} &= (-1)^{s}\cos(0)\epsilon^{2N-k-1} = (-1)^{s}\epsilon^{2N-2s-2} \\
&= \sum_{t=0}^{N} \frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}}\epsilon^{2t} + O(\epsilon^{2N+1}).
\end{aligned}
$$

So we get : for $t \neq N-s-1$, $\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}} = 0$; for $t = N-s-1$, $\frac{1}{(2t)!}\frac{\partial^{2t}\varphi_{k1}}{\partial\epsilon^{2t}} = (-1)^{s}$. It can be rewritten as

Proposition 4.7

$$
\begin{aligned}
&0 \le t \le N-1,\ 0 \le s \le N-1, \\
n_{2s+1,t+1} = 0 \ \textit{if}\ t < N-s-1, \quad &n_{2s+1,t+1} = (-1)^{s} \ \textit{if}\ t = N-s-1. \qquad (34)
\end{aligned}
$$

d. $l'_{k,N+1}$ for k even, $k = 2s$.

$$
l'_{k1} = (-1)^{s}\sin(0)\epsilon^{2N-k-1} = 0
$$

So we get

Proposition 4.8

$$0 \leq t \leq N-1, \ 1 \leq s \leq N,$$
$$n_{2s,t+1} = 0. \tag{35}$$

Then $A_1(N) := (\prod_{l=1}^{N-1}(2l)!)^{-2} \det((d_{jk})_{j,k\in[1,2N]})_{(\tilde{a_j}=\tilde{b_j}=0,\, 1\leq j\leq N-1,\, x=0,\, t=0)}$ can be written as $\det((\tilde{d}_{ij})_{i,j\in[1,2N]})$ defined by

$$\begin{vmatrix}
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & (-1)^0 \\
(-1)^0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & (-1)^1 & 0 \\
0 & (-1)^1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & (-1)^{N-2} & 0 & 0 & (-1)^{N-2} & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & (-1)^{N-1} & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & (-1)^{N-1} & 0 & 0 & \ldots & 0 & 0
\end{vmatrix} \tag{36}$$

In the previous determinant $A_1(N)$, on each row, only one term is non equal to 0. Then we expand along the last row and again along the last row of the new determinant; we obtain the recurrence relation : $A_1(N) = -A_1(N-1)$. As $A_1(1) = -1$, this relation proves that $A_1(N) = (-1)^N$.
3. Then we can evaluate the absolute value of the quotient $|v_0(0,0)| = |\frac{A3}{A1}| = |\frac{2N+1}{(-1)^N}| = 2N+1$.
The maximum of amplitude of the modulus of the AP breather of order N is equal to $2N+1$. $\square$

5 An other simpler representation of families of quasi-rational solutions depending on $2N-2$ parameters of the NLS equation

We saw in previous section that solutions of NLS equation given by (11) can be written in function uniquely of terms γ. We recall that the terms γ_j are given by $\gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}$, $\gamma_{N+j} = \frac{1}{\gamma_j}$, $1 \leq j \leq N$; λ_j is an arbitrary real parameter such that $0 < \lambda_j < 1$, $\lambda_{N+j} = -\lambda_j$, $1 \leq j \leq N$.
We can rewrite the result given in (11) in a simplest formulation as follows :

Theorem 5.1 *The function v defined by*

$$v(x,t) = \exp(2it - i\varphi) \times \frac{\det((f^{(3)}_{jk})_{j,k\in[1,2N]})}{\det((f^{(1)}_{jk})_{j,k\in[1,2N]})} \tag{37}$$

is a quasi-rational solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0,$$

depending on $2N-2$ *real parameters* $\tilde{a}_j$, $\tilde{b}_j$, $1 \leq j \leq N-1$ *where*

$$f_{jk}^{(r)} = \frac{\partial^{2(k-1)}}{\partial \epsilon^{2(k-1)}} \left(\gamma^{4j-1} \sin\left[\frac{2\gamma}{1+\gamma^2} x + 4i \frac{\gamma(1-\gamma^2)}{(1+\gamma^2)^2} t - i\frac{r-1}{2} \ln \frac{\gamma-i}{\gamma+i} + \sum_{l=1}^{N-1} (\tilde{a}_l + i\tilde{b}_l) \epsilon^{2l+1} + (j-1)\frac{\pi}{2} \right] \right)_{(\epsilon=0)},$$
$$f_{jN+k}^{(r)} = \frac{\partial^{2(k-1)}}{\partial \epsilon^{2(k-1)}} \left(\gamma^{2N-4j-1} \cos\left[\frac{2\gamma}{1+\gamma^2} x - 4i \frac{\gamma(1-\gamma^2)}{(1+\gamma^2)^2} t + i\frac{r-1}{2} \ln \frac{\gamma-i}{\gamma+i} + \sum_{l=1}^{N-1} (\tilde{a}_l - i\tilde{b}_l) \epsilon^{2l+1} + (j-1)\frac{\pi}{2} \right] \right)_{(\epsilon=0)},$$
$$1 \leq k \leq N, \quad 1 \leq j \leq 2N, \quad r \in \{1;3\}, \quad \epsilon \in]0;1[, \quad \gamma = \epsilon(1-\epsilon^2)^{1/2}.$$

Proof : It is sufficient to use the formulation given in the previous section which needs four types of functions and to take $\lambda = 1 - 2\epsilon^2$.
□

Remark 5.1 *In the previous theorem, the expression* $\frac{\partial^0}{\partial \epsilon^0} f(x)$ *means* $f(x)$*.*

6 Quasi-rational solutions of order 8 with 14 parameters

6.1 Deformations of the AP_8 breather

We have already constructed the deformations of the AP_N breathers with $2N-2$ in a series of papers for the cases $N = 3$ to 7 [14, 15, 16, 17, 18]. We only construct the patterns corresponding to the case $N = 8$ in order not to weight down the paper.
All the study conducted in this article and figures in particular show that the solutions obtained with this method are completely localized at the same time in time and space.
In this case we obtain a family of solutions depending on 14 parameters. The analytical expression and their two-parameters deformations can be found in [10], the 14-deformations are found, but too monstrous to be published.
We give plots of deformation of the AP breather of order 8 by variations of one parameter (we only present here the case of parameters $a_k \neq 0$). In particular, we recover triangles as given in 2012 in [31] and circular structures as already presented first in 2013 [32].

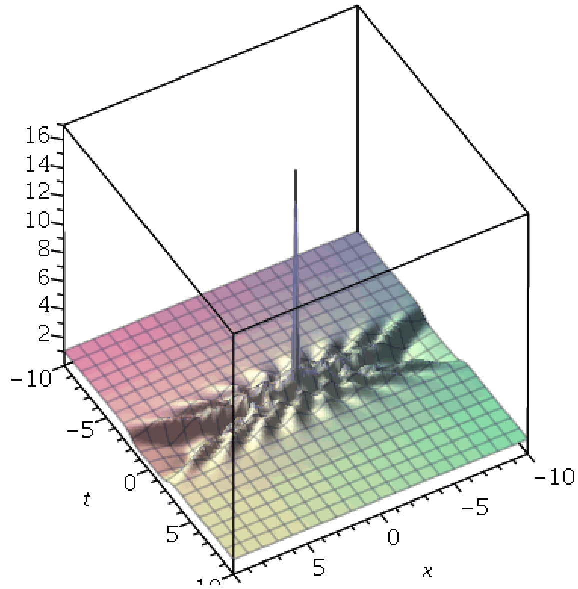

Figure 1: Solution of NLS, N=8; all parameters equal to 0, the AP_8 breather.

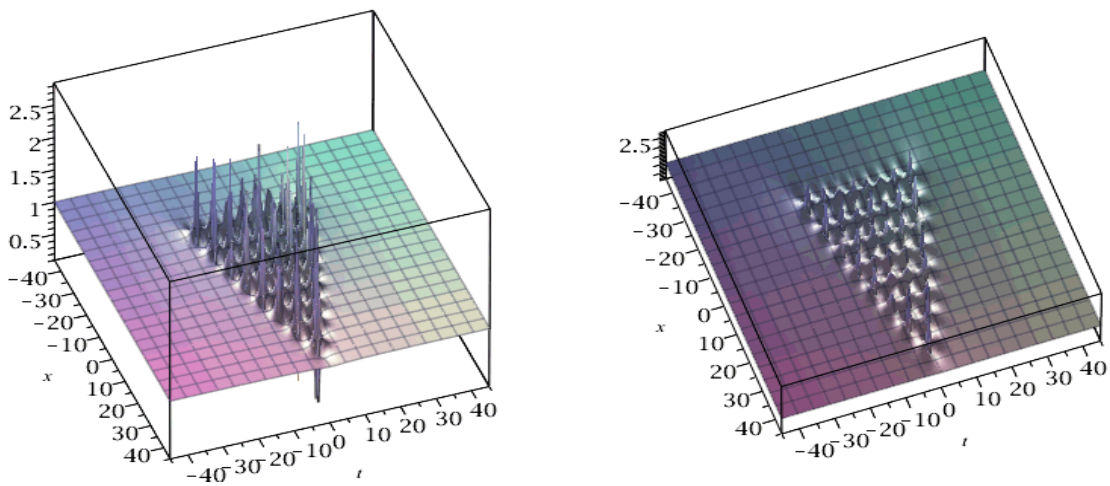

Figure 2: Solution of NLS, N=8; $a_1 = 10^2$: a regular triangle with 36 peaks; right : sight of top.

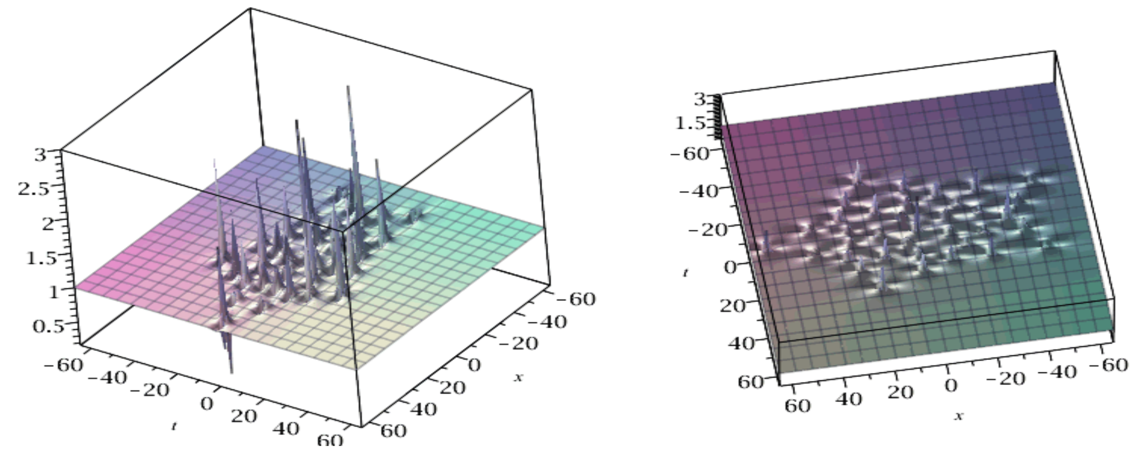

Figure 3: Solution of NLS, N=8; $\tilde{a}_2 = 10^6$, 7 rings with 5 peaks on each of them with in the center one peak; right : sight of top.

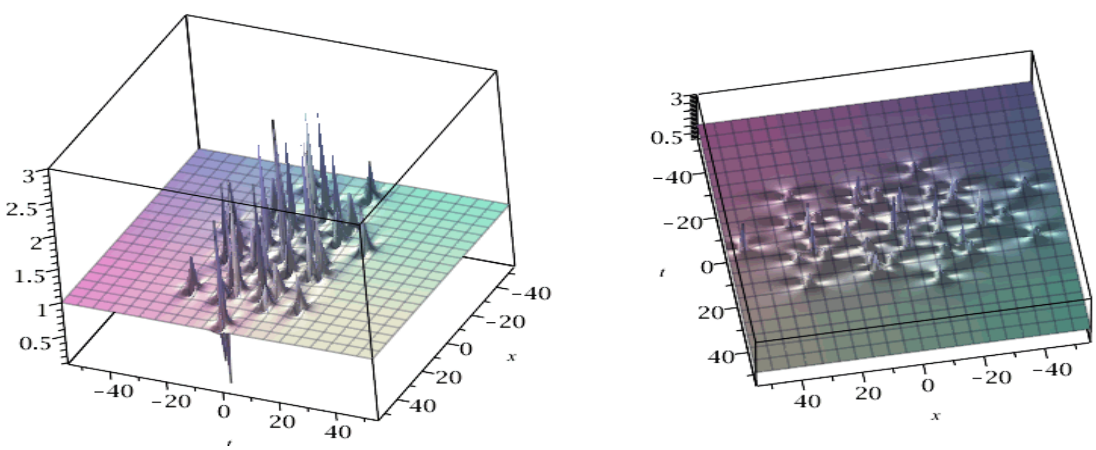

Figure 4: Solution of NLS, N=8; $\tilde{a}_3 = 10^8$, 5 rings with 7 peaks on each of them with a central peak; right : sight of top.

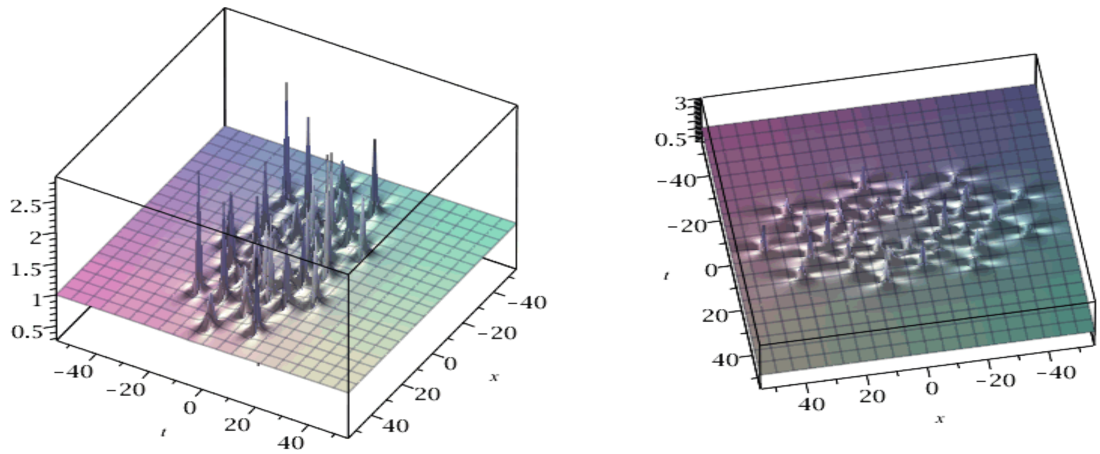

Figure 5: Solution of NLS, N=8; $\tilde{a}_4 = 10^{10}$, 4 rings with 9 peaks on each of them without central peak; right : sight of top.

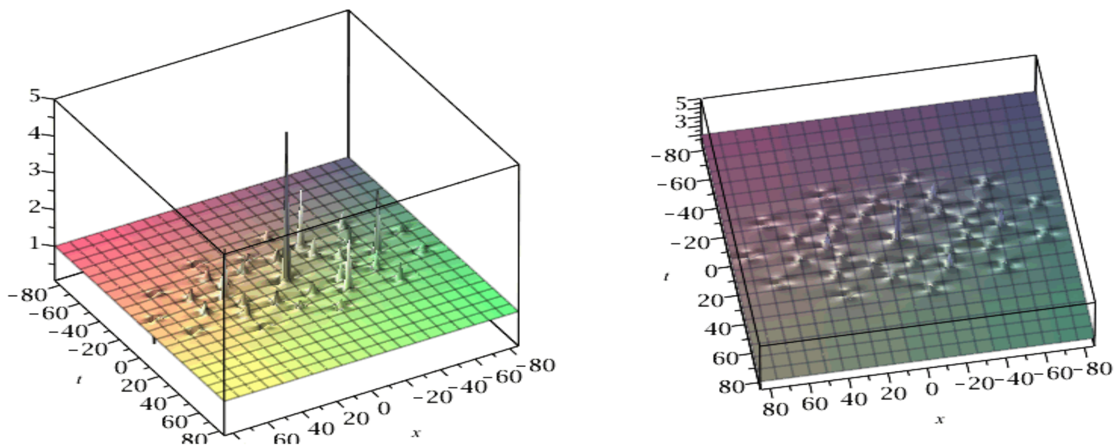

Figure 6: Solution of NLS, N=8; $\tilde{a}_5 = 10^{15}$; 3 rings of 11 peaks with in the center P_2; right : sight of top.

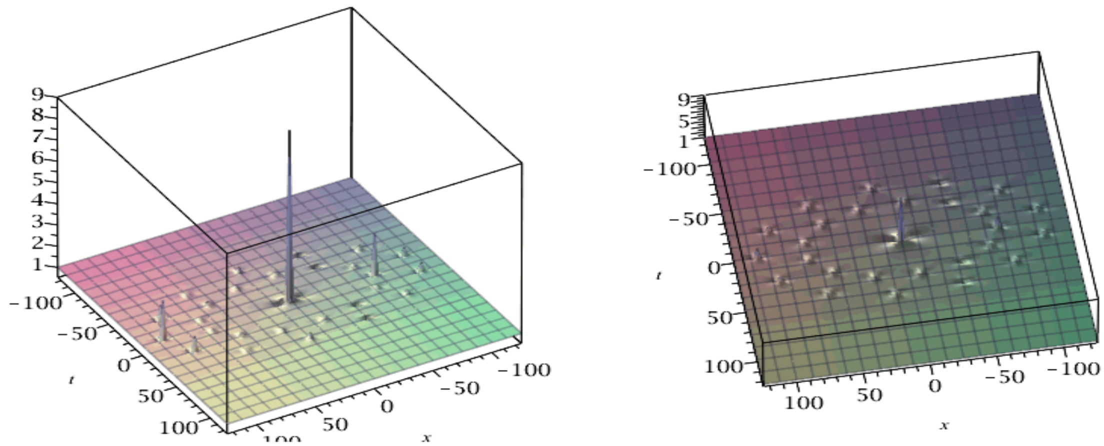

Figure 7: Solution of NLS, N=8; $\tilde{a}_6 = 10^{20}$, 2 rings with 13 peaks and in the center P_4; right : sight of top.

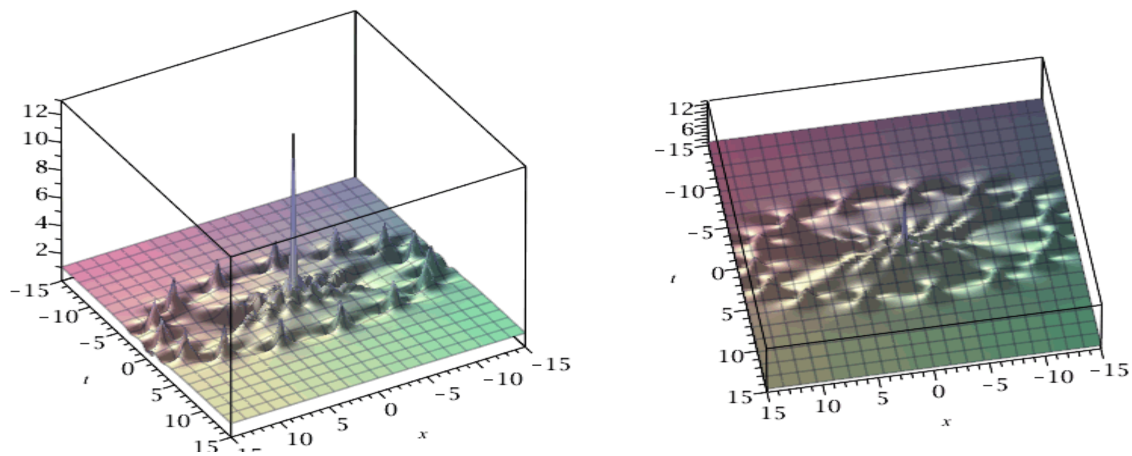

Figure 8: Solution of NLS, N=8; $\tilde{a}_7 = 10^{10}$, one ring with 15 peaks and in the center P_6; right : sight of top.

These structures appear as soon as a_k or b_k exceed a certain value. These structures most sensitive to the variation of the parameters a_k or b_k are those obtained for $k = 1$; a formation of a triangle appears as soon as a_1 or b_1 is greater than 100. The more k increases, the more the structures appear for large values of the parameter; for example for $k = 7$, the structures appear clearly for a_7 or b_7 about 10^8. The heights of the peaks increase as k increases by 1 to 7; for $k = 1$, the maximum height is about 3, to increase until $k = 7$ where it becomes about 13. The dissipation of the structure is all the more slow as k is large.

6.2 Asymptotic behavior

In all these plots, in the case of order N, we see when only one of the parameters $\tilde{a}_{N-1}$ or $\tilde{b}_{N-1}$ is not equal to 0 and one of these parameters tends to infinity the appearance of the AP breather of order $N-2$. This observation was first pointed out by Matveev. In fact, the computations show when the parameters $\tilde{a}_{N-1}$ or $\tilde{b}_{N-1}$ tend to infinity, for order N, the appearance as factor of $\tilde{a}^2_{N-1}$ or $\tilde{a}^2_{N-1}$ of the analytic expression of the Akhmediev-Peregrine of order $N-2$. This fact is shown by computations but these results are too long to be presented in this paper.

7 Conclusion

We have constructed here new representations of solutions of the NLS focusing equation. These solutions appear as $2N-2$-parameters deformations of the AP breather of order N.
A subset of the solutions was built. With this subset, a proof that the maximum of the modulus of the breather of order N is equal to $2N+1$ was given. This result was conjectured by Akhmediev for the first time in 2010 [4]; we also find this conjecture in the works of many authors, in particular we can mention Matveev [8], He [28], Yang [34].
Akhmediev et al. gave first the proof of this result for $N = 1$ to $N = 4$ in the case of solutions without parameters in 2010 [4]. Here we give another approach to the proof in the case of solutions depending on $2N-2$ parameters at order N, for any nonnegative integer N.
In the case of the variation of one parameter for $N = 8$, we obtain different

types of configurations with a maximum of 36 peaks. For $\tilde{a}_1 \neq 0$ or $\tilde{b}_1 \neq 0$ we obtain triangles with a maximum of 36 peaks; for $\tilde{a}_2 \neq 0$ or $\tilde{b}_2 \neq 0$, we have 5 rings with respectively 5, 10, 5, 10, 5 peaks with one peak in the center. For $\tilde{a}_3 \neq 0$ or $\tilde{b}_3 \neq 0$, we obtain 5 rings with 7 peaks on each of them with a central peak. For $\tilde{a}_4 \neq 0$ or $\tilde{b}_4 \neq 0$, we have 4 rings with 9 peaks on each of them without central peak. For $\tilde{a}_5 \neq 0$ or $\tilde{b}_5 \neq 0$, we have 3 rings of 11 peaks with in the center the Akhmediev-Peregrine of order 2. For $\tilde{a}_6 \neq 0$ or $\tilde{b}_6 \neq 0$, we have 2 rings with 13 peaks and in the center the AP breather of order 4. For $\tilde{a}_7 \neq 0$ or $\tilde{b}_7 \neq 0$, we have one ring with 15 peaks and in the center the AP breather of order 6.
Many studies have been done these last years, but this is the first time that this study for order 8 is realized with 14 parameters. The expressions of the polynomials in x and t are found; they are too extensive to be published. We have only given the plots in order to illustrate the deformations of the solutions.
Moreover, we recover the asymptotic behavior of the solutions in the cases where a_7 or b_7 are not equal to zero; in the case $N = 8$, for large values of these parameters, we see in the center of the ring formed of $2N - 1$ peaks, the appearance of breather of order $N - 2$.
This last result already mentioned by Akhmediev, He, should be proved in the next years, like many other properties about the appearance of the peaks depending on the different choices of parameters.
The solutions given in this article are exact solutions to NLS equation. By construction, according to the theory, total energy is preserved and independent of time for this equation. These solutions represent quasi rational solutions of order N fixed by the condition which its modulus tends towards 1 when x or t tends to infinite and its highest maximum is localized at the point $(x = 0; t = 0)$. In particular Akhmediev-Peregine breather AP_N is distinguished by the fact that $AP_N(0; 0) = 2N+1$, and that among the previous class, the solution which has a larger modulus. The solutions constructed in the article are deformations of Akhmediev-Peregrine breather of order 8, and it is well-known that this one is a solution to the NLS equation and does not contradict in no case the principle of conservation of total energy and total moment.
Moreover, it is important to stress on the fact that the modulus of these solutions tend towards 1 when x or t tends towards infinite : the modulus of these solutions is a quotient of two polynomials of degree 72 in x and t; each coefficient of x^{72} and t^{72} of these polynomials representing the solutions

is equal in the numerator and the denominator what proves the result of the asymptotic behavior of the solutions.

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